Spectral Degeneracies in the Totally Asymmetric Exclusion Process

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Abstract

We study the spectrum of the Markov matrix of the totally asymmetric exclusion process (TASEP) on a one-dimensional periodic lattice at *arbitrary* filling. Although the system does not possess obvious symmetries except translation invariance, the spectrum presents many multiplets with degeneracies of high order. This behaviour is explained by a hidden symmetry property of the Bethe Ansatz. Combinatorial formulae for the orders of degeneracy and the corresponding number of multiplets are derived and compared with numerical results obtained from exact diagonalisation of small size systems. This unexpected structure of the TASEP spectrum suggests the existence of an underlying large invariance group.

Keywords: ASEP, Markov matrix, Bethe Ansatz, Symmetries.

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1 Introduction

The asymmetric simple exclusion process (ASEP) is a driven lattice gas model in which particles interact by hard core exclusion. This simple system has been introduced as a building block for models of hopping conductivity, motion of RNA templates, traffic flow and surface growth. From a theoretical point of view, the ASEP plays a fundamental role in the study of

non-equilibrium processes: many exact results have been obtained concerning one-dimensional phase transitions (Derrida et al. 1993), phase segregation (Evans et al. 1998), large deviations functions and fluctuations far from equilibrium (Derrida et al. 2003). For a review, see Derrida (1998), Schütz (2001). In the absence of a driving field, the *symmetric* exclusion process can be mapped into the Heisenberg spin chain. The asymmetry due to a non-zero external driving field breaks the left/right symmetry and the ASEP is equivalent to a non-Hermitian spin chain of the XXZ type. The ASEP can also be mapped into a two-dimensional six-vertex model at equilibrium. These mappings allow to use the methods of integrable systems, such as the Bethe Ansatz (Dhar 1987, Gwa and Spohn 1992, Kim 1995).

In a recent article (Golinelli and Mallick 2004), we carried out a spectral study of the Markov matrix of the exclusion process on a periodic lattice. At half-filling the system is invariant under charge conjugation combined with reflection in addition to being translation invariant. We showed that these symmetries predict the existence of singlets and doublets in the spectrum. However, for the *totally* asymmetric simple exclusion process (TASEP) the spectral structure is much richer. We observed numerically that unexpected degeneracies of higher order exist and become generic as the system size increases. We explained, in an heuristic manner, the existence of these degeneracies by using the fact that some of the solutions of the Bethe equations appear in pairs with opposite values.

In the present work, we generalize our previous study to the TASEP at *arbitrary filling*. We perform an exhaustive analysis of the spectral degeneracies of the TASEP on a periodic lattice. The orders of degeneracies observed and the number of multiplets with a given degeneracy depend on commensurability relations between the number of sites and the number of particles. Although the Bethe equations do not exhibit any obvious symmetry, we find that they possess an invariance under exchange of roots that allows to predict combinatorial formulae for the orders of degeneracies and the number of multiplets of a given order of degeneracy. Our formulae are confirmed by direct numerical diagonalisation of small size systems. This peculiar structure of the TASEP spectrum suggests the existence of some underlying symmetries of the model that may shed light on its remarkable combinatorial properties.

The outline of this article is as follows. In section 2, the definition and basic properties of the TASEP are recalled. In section 3, we present a self-contained derivation of the Bethe equations based upon the fact that the Bethe wave function is a determinant for the TASEP model. The symmetries of the Bethe equations are studied in section 4 and combinatorial expressions for the degeneracies are derived in section 5. In section 6, we discuss the behaviour of large size systems and present numerical results. Concluding

remarks appear in section 7. In the Appendix, the geometrical setting of the roots in the complex plane is briefly described.

2 The TASEP model

The simple exclusion process is a stochastic process in which particles hop on a lattice and respect the *exclusion rule* that forbids two or more particles per site. On a one-dimensional lattice, this rule prohibits overtaking between particles.

In this article we consider a periodic 1-d lattice of length L, i.e., a ring where sites i and $i+L$ are identical. The system is closed and the number N of particles is fixed with $N \leq L$. The *filling* (or density) is given by $\rho = N/L$.

The particles evolve with the following dynamics rule: during the time interval $[t, t+dt]$, a particle on a site i jumps with probability dt to the neighbouring site $i+1$ if it is vacant. As the jumps are allowed in only one direction, the model considered is the *totally asymmetric* exclusion process (TASEP).

A configuration $\mathcal{C}$ of the system is characterised by the list of the N occupied sites amongst the L available sites. The total number of configurations is therefore

$$\Omega = \binom{L}{N} = \frac{L!}{N!(L-N)!}\,. \tag{1}$$

Let $\psi_t(\mathcal{C})$ be the probability that the configuration of the system at time t is $\mathcal{C}$. As the TASEP is a continuous-time Markov (i.e., memoryless) process, the Ω-dimensional vector ψ_t evolves according to the *master equation*

$$\frac{d\psi_t}{dt} = M\psi_t\,, \tag{2}$$

where M is the $\Omega \times \Omega$ Markov matrix. For $\mathcal{C} \neq \mathcal{C}'$, the term $M(\mathcal{C}', \mathcal{C})$ represents the transition rate from $\mathcal{C}$ to $\mathcal{C}'$: it is equal to 1 if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by an allowed jump of one particle, and is 0 otherwise. The element $-M(\mathcal{C}, \mathcal{C})$ is equal to the number of allowed jumps from $\mathcal{C}$. Thus, the sums columns of the Markov matrix vanish and the total probability is conserved. For the exclusion process on a periodic lattice, the sums over lines of M also vanish: the stationary probability (which corresponds to the eigenvalue 0) is thus uniform: $\psi(\mathcal{C}) = 1/\Omega$.

As the dynamics is ergodic and aperiodic, M satisfies the conditions of the Perron-Frobenius theorem: the eigenvalue 0 is non-degenerate and all the other eigenvalues have strictly negative real parts (equal to the inverse of

the relaxation times). As M is a real non-symmetric matrix, the eigenvalues are either real numbers or complex conjugate pairs.

In this work, we shall investigate the spectral degeneracies, i.e., equalities amongst the eigenvalues of the Markov matrix.

3 Derivation of the Bethe equations

Since the work of Dhar (1987), it is known that the *Bethe Ansatz* can be applied to the ASEP. In this section, we give a self-contained derivation of the Bethe equations for the particular case of the TASEP, much simpler than that of the generic ASEP (Gwa and Spohn 1992).

A configuration $\mathcal{C}$ will be represented by the sequence $(x_1, x_2, \ldots, x_N)$, the integers x_i being the positions of the particles with

$$1 \leq x_1 < x_2 < \ldots < x_N \leq L\,. \tag{3}$$

The idea of the Bethe *Ansatz* consists in writing the eigenvectors ψ of the Markov matrix as linear combinations of plane waves (see, e.g., Gaudin 1983). In fact, the Bethe wave function ψ of the TASEP is a determinant (Gaudin and Pasquier 2004). We therefore define ψ as

$$\psi(x_1, \ldots, x_N) = \det(R)\,, \tag{4}$$

where R is a $N \times N$ matrix with elements

$$R(i,j) = \frac{z_i^{x_j}}{(1-z_i)^j} \quad \text{for } 1 \leq i,j \leq N\,, \tag{5}$$

$(z_1, \ldots, z_N)$ being N given complex numbers. If we assume that ψ is of this form and that it is an eigenvector of M, the z_i's then must satisfy some conditions, the *Bethe equations*, that we now re-derive.

We first show that ψ defined by equations (4, 5) satisfies two identities which are valid for any values of z_i and of x_i, even without imposing the ordering given in equation (3). The first identity is

$$E\psi(x_1, \ldots, x_N) = \sum_{k=1}^{N} [\psi(x_1, \ldots, x_k - 1, \ldots, x_N) - \psi(x_1, \ldots, x_k, \ldots, x_N)]\,, \tag{6}$$

for any $(x_1, \ldots, x_N)$ and $(z_1, \ldots, z_N)$, and where E is given by

$$E = -N + \sum_{i=1}^{N} 1/z_i\,. \tag{7}$$

Equation (6) is obtained by writing

$$\begin{aligned}&\psi(x_1,\dots,x_k-1,\dots,x_N)-\psi(x_1,\dots,x_k,\dots,x_N)=\\&\quad\det\left(R(i,1),\dots,\left(\frac{1}{z_i}-1\right)R(i,k),\dots,R(i,N)\right).\end{aligned}\tag{8}$$

This determinant is similar to $\det(R)$ except for the k-th column. Expanding this determinant over all permutations of $\{1,\dots,N\}$ and performing the sum over $k=1,\dots,N$ leads to equations (6, 7).

The second identity valid for any $(z_1,\dots,z_N)$ and any $(x_1,\dots,x_N)$ with $x_{k-1}=x_k$ (two particles collision), is

$$\psi(x_1,\dots,x_k,x_k,\dots,x_n)-\psi(x_1,\dots,x_k,x_k+1,\dots,x_n)=0\,.\tag{9}$$

The left hand side of equation (9) can be written as $\det(\tilde{R})$ where $\tilde{R}$ is a matrix that is identical to R but for its k-th column that is given by

$$\tilde{R}(i,k)=\frac{z_i^{x_k}-z_i^{x_k+1}}{(1-z_i)^k}=\frac{z_i^{x_k}}{(1-z_i)^{k-1}}=R(i,k-1)=\tilde{R}(i,k-1)\tag{10}$$

The $(k-1)$-th and the k-th columns of $\tilde{R}$ are equal and, therefore, $\det(\tilde{R})=0$. This proves equation (9).

The eigenvalue equation, $E\psi=M\psi$, is written as equation (6) except that the sum is restricted to the allowed jumps of particles, i.e., to the indices k such that $x_{k-1}+1<x_k$. However, in equation (6), the terms with $x_{k-1}+1=x_k$ vanish thanks to equation (9). Thus equation (6) is identical to the eigenvalue equation if the eigenvector has the form assumed in equations (4, 5).

The function ψ must also satisfy periodic boundary conditions

$$\psi(x_1,x_2,\dots,x_n)=\psi(x_2,\dots,x_n,x_1+L)\,.\tag{11}$$

The periodic conditions are the ones that quantify the eigenvalues by imposing a set of equations on the z_i's, the Bethe equations. Denoting by i and j the generic line and column indices of the matrices, respectively, we have

$$\psi(x_2,\dots,x_N,x_1+L)=\det\left(\frac{z_i^{x_2}}{1-z_i},\dots,\frac{z_i^{x_{j+1}}}{(1-z_i)^j},\dots,\frac{z_i^{x_1+L}}{(1-z_i)^N}\right).\tag{12}$$

By cyclic permutation of the columns, we obtain

$$\begin{aligned}&\psi(x_2,\dots,x_N,x_1+L)\\=\;&(-1)^{N-1}\det\left(\frac{z_i^{x_1+L}}{(1-z_i)^N},\frac{z_i^{x_2}}{1-z_i},\dots,\frac{z_i^{x_j}}{(1-z_i)^{j-1}},\dots\right)\end{aligned}$$

$$= (-1)^{N-1} \det\left(\frac{z_i^L}{(1-z_i)^{N-1}} R(i,1), \ldots, (1-z_i)\, R(i,j), \ldots \right)$$
$$= (-1)^{N-1} \prod_{k=1}^{N} (1-z_k) \quad \det\left(\frac{z_i^L}{(1-z_i)^{N}} R(i,1), \ldots, R(i,j), \ldots \right) . \quad (13)$$

This last term is equal to $\psi(x_1, x_2, \ldots, x_n) = \det(R)$ if $z_1, \ldots, z_N$ are solutions of the N Bethe equations

$$(z_i - 1)^N z_i^{-L} = -\prod_{k=1}^{N} (1 - z_k) \quad \text{for} \quad i = 1, \ldots, N\,. \quad (14)$$

The vector ψ defined by equations (4, 5) is then an eigenvector of the Markov matrix M with eigenvalue E given by equation (7).

The uniform stationary probability with $E = 0$ corresponds to the very special solution where all the $z_i = 1$. For all the other solutions of the Bethe equations, the z_i's are distinct and are different from 1; hence the determinant $\det(R)$ does not vanish.

4 Symmetries of the Bethe equations

In this section we show that the Bethe equations (14) have certain solutions that are distinct but lead to the same eigenvalue E.

Following Gwa and Spohn (1992), we introduce new variables $\tilde{z}_i = 2/z_i - 1$ in the Bethe equations (14) which then become

$$(1 - \tilde{z}_i)^N \, (1 + \tilde{z}_i)^{L-N} = -2^L \prod_{k=1}^{N} \frac{\tilde{z}_k - 1}{\tilde{z}_k + 1} \quad \text{for} \quad i = 1, \ldots, N\,. \quad (15)$$

The corresponding eigenvalue E is given by

$$2E = -N + \sum_{k=1}^{N} \tilde{z}_k\,. \quad (16)$$

We remark that the right hand side of equation (15) is independent of i. We shall analyse equation (15) in three steps. Firstly, we consider the polynomial equation of degree L for a given complex parameter Y,

$$(1 - Z)^N \, (1 + Z)^{L-N} = Y\,. \quad (17)$$

We call $(Z_1, \ldots, Z_L)$ the roots of this polynomial. In the Appendix, we explain how the Z_i's can be labelled so that for a given i, Z_i is an analytic

function of Y in the complex plane with a branch cut along the real semi-axis $[0, +\infty)$. Secondly, we choose a set $c = \{c_1, \ldots, c_N\}$ of N distinct indices among $\{1, \ldots, L\}$. The number of possible sets c is precisely Ω, the total number of configurations (equation (1)). Finally, for a given choice set c, we define a function of Y

$$A_c(Y) = -2^L \prod_{k=1}^{N} \frac{Z_{c_k} - 1}{Z_{c_k} + 1} \,. \tag{18}$$

The Bethe equations (15) are now equivalent to the self-consistency equation

$$A_c(Y) = Y \,. \tag{19}$$

For a given choice set c and a root Y_c of the last equation, the Z_k's are determined by equation (17). The solutions of the Bethe equations are then given by $\tilde{z_k} = Z_{c_k}$. The corresponding eigenvalue E_c is obtained from equation (16)

$$2E_c = -N + \sum_{k=1}^{N} Z_{c_k} \,. \tag{20}$$

The eigenvector ψ is given by equations (4, 5), using $z_i = 2/(\tilde{z_i} + 1)$.

In order to understand the origin of the spectral degeneracies, we must consider the case where L (the number of sites) and N (the number of particles) are not relatively prime. We define the integers p, n and l as follows

$$p = \gcd(L, N) \;, \quad L = pl \;, \quad N = pn \;. \tag{21}$$

Let $(y_1, \ldots, y_p)$ be the p-th roots of Y (i.e., $y_k^p = Y$) labelled as $0 \leq \arg(y_1) < \ldots < \arg(y_p) < 2\pi$. Equation (17) is thus equivalent to the p polynomial equations of degree l

$$Q_k(Z) = (1 - Z)^n \, (1 + Z)^{l-n} - y_k = 0 \quad \text{for} \quad k = 1, \ldots, p \,. \tag{22}$$

Thus the set $\{Z_1, \ldots, Z_L\}$ of the L solutions of equation (17) is made up of p *packages*, the k-th package being constituted by the l solutions of $Q_k(Z) = 0$. Let us call $\mathcal{P}_k$ the set of indices of the k-th package: in other words the solutions of $Q_k(Z) = 0$ are the Z_i with $i \in \mathcal{P}_k$. The labelling of the Z_i described in the Appendix shows explicitly that

$$\mathcal{P}_k = \{k, k + p, k + 2p, \ldots, k + L - p\} \,, \tag{23}$$

i.e., $\mathcal{P}_k$ contains the indices i such that $i = k$ modulo p.

For any given package $\mathcal{P}_k$, we have the fundamental equations

$$\sum_{i\in\mathcal{P}_k} Z_i = 2n - l\,, \tag{24}$$

$$\prod_{i\in\mathcal{P}_k} \frac{Z_i - 1}{Z_i + 1} = 1\,. \tag{25}$$

We emphasize that the right hand sides are independent of k. These identities are derived as follows. We have $(-1)^n Q_k(Z) = \prod_{i\in\mathcal{P}_k}(Z - Z_i)$ because the Z_i's with $i \in \mathcal{P}_k$ are the roots of the polynomial $Q_k(Z)$. The evaluation of the coefficient of Z^{l-1} leads to the equation (24) except when $l = 1$. Similarly, the evaluation of $Q_k(1)/Q_k(-1)$ yields equation (25) except when $n = 0$ or $n = l$. These exceptions correspond to a trivial model which is either empty, $N = 0$, or full, $N = L$, (the spectrum of M is then reduced to the single eigenvalue $\{0\}$). In the following, we assume that $0 < N < L$ and so equations (24, 25) are true.

We now consider a solution Y_c of the Bethe equation (19) associated with a given choice set c. Assume that there exists a package $\mathcal{P}_f$ such that c contains $\mathcal{P}_f$ (i.e., $\mathcal{P}_f \subset c$) and that there exists another package $\mathcal{P}_e$ such that c is disjoint from $\mathcal{P}_e$ (i.e., $\mathcal{P}_e \cap c = \emptyset$). We define a new choice set c' of N indices by exchanging the package $\mathcal{P}_f$ with $\mathcal{P}_e$, i.e., $c' = (c/\mathcal{P}_f)\cup\mathcal{P}_e$. We now show that the eigenvalues associated with c and c' are equal. Indeed, because of equation (25), the contribution of $\mathcal{P}_f$ in $A_c(Y)$ and the contribution of $\mathcal{P}_e$ in $A_{c'}(Y)$ are both equal to 1 and therefore $A_c(Y) = A_{c'}(Y)$, the other packages contained in c and c' being the same. Thus Y_c is also a solution of the Bethe equation associated with the set c', i.e., $A_{c'}(Y_c) = Y_c$. Besides, thanks to equation (24), we notice that the contribution of $\mathcal{P}_f$ to the eigenvalue E_c is equal to the contribution of $\mathcal{P}_e$ to $E_{c'}$. Thus, because the other packages contained in c and c' are the same, we conclude from equation (20) that $E_c = E_{c'}$. However, the corresponding eigenvectors are different: some of the chosen z_i's being different for c and c', the functions $\psi_c(x_1, \ldots, x_N)$ and $\psi_{c'}(x_1, \ldots, x_N)$ are not equal for the Ω different configurations. We have thus obtained a degenerate eigenvalue $E_c = E_{c'}$ associated with two different sets c and c'.

To summarise, an eigenvalue corresponding to a choice set c is degenerate if there exists at least one package $\mathcal{P}_f$ entirely contained in c and at least one package $\mathcal{P}_e$ that does not intersect with c. The fundamental reason is that a full package of Z_i's does not contribute to the Bethe equations and adds up to a constant contribution in the eigenvalue formula. Therefore exchanging the packages $\mathcal{P}_f$ and $\mathcal{P}_e$ leads to the same eigenvalue but not to the same eigenvector and results in degeneracies in the spectrum.

As the size of the packages is l, we note that degeneracies are possible only if $l \leq N \leq L - l$ with $l = L/\gcd(L, N)$.

5 Combinatorial formulae for the degeneracies

We shall now enumerate the degeneracies in the spectrum of the Markov matrix M of the TASEP with N particles evolving on a ring of L sites. We assume the following *one-to-one hypothesis*: for each choice set c (among the Ω possible sets), the self-consistency equation (19) has a unique solution Y_c that provides the eigenvalue E_c and the eigenvector ψ_c. We further assume that these eigenvectors are linearly independent; this hypothesis, combined with the fact that the dimension of the configuration space is Ω, implies that the Bethe equations provide a complete basis of eigenvectors. We emphasize that the one-to-one hypothesis is stronger than the assumption that the Bethe basis is complete. We have observed numerically on small size systems that the functions $A_c(Y)$ are usually contraction mappings (which would imply the one-to-one hypothesis), but we have not succeeded yet to obtain a rigorous proof. However we will see that this one-to-one hypothesis allows to count the degeneracies and that the results are in perfect agreement with numerical diagonalisations. Thus we are convinced that this hypothesis, or a weaker hypothesis with the same consequences, is true and that it should be possible to prove it.

With this one-to-one hypothesis, counting degeneracies becomes merely an exercise in combinatorics. We first introduce some notations. We recall (see Section 4) that $c = \{c_1, \ldots, c_N\}$ is a set of N integers chosen amongst $\{1, \ldots, L\}$; moreover $\{1, \ldots, L\}$ is partitioned in p packages $\mathcal{P}_1, \ldots, \mathcal{P}_p$, each containing $l = L/p$ integers with $p = \gcd(L, N)$ (see equations(21–23)). For a given set c and for $0 \leq i \leq l$, we denote by a_i the number of packages $\mathcal{P}_k$ with i elements in c (i.e., such that $\mathcal{P}_k \cap c$ has i elements). Thus a_0 is the number of packages that do not intersect c: such packages will be referred to as 'empty' packages. The number of the 'full' packages (i.e., entirely included in c) is a_l. We call *partial* packages those that are neither empty nor full. Following this definition, we have

$$a_i \geq 0 \quad , \quad \sum_{i=0}^{l} a_i = p \quad , \quad \sum_{i=0}^{l} i\, a_i = N \,, \tag{26}$$

p being the total number of packages and N the cardinality of the set c. We call $a = (a_0, a_1, \ldots, a_l)$ an *admissible partition* if it satisfies the relations in

equation (26). Equivalently an admissible partition corresponds to a partition of the integer N in which each term is $\leq l$ and which contains at most p terms.

The total number $\omega(a)$ of choice sets c corresponding to a given admissible partition a is

$$\omega(a) = \frac{p!}{a_0!a_1!a_2!\dots a_l!}\binom{l}{0}^{a_0}\binom{l}{1}^{a_1}\binom{l}{2}^{a_2}\dots\binom{l}{l}^{a_l} = p!\prod_{i=0}^{l}\frac{1}{a_i!}\binom{l}{i}^{a_i}. \quad (27)$$

In this equation, the first factor enumerates the number of choices for each type of packages among the p available packages. The factors of the type $\binom{l}{i}^{a_i}$ give the number of choices of i elements among l for each of the a_i packages. According to the one-to-one hypothesis, $\omega(a)$ represents also the number of eigenvalues associated with the admissible partition a.

We have shown at the end of the previous section that two sets c and c' provide a degenerate eigenvalue $E_c = E_{c'}$ if they are built from the same partial packages and differ only by the selected empty and full packages. Thus the eigenvalue E_c is $d(a)$ times degenerate with

$$d(a) = \binom{a_0 + a_l}{a_l}. \quad (28)$$

This relation is obtained by enumerating all the choice sets c' obtained from c by keeping the partial packages unchanged and choosing a_l full packages from the remaining $a_l + a_0$ packages. We remark that c and c' correspond to the same admissible partition a. We also emphasize that the degeneracy order depends only on a and not on the precise choice set c. To resume, a single order of degeneracy $d(a)$ is associated with the admissible partition a.

Consequently, the $\omega(a)$ eigenvalues corresponding to the admissible partition a form $m(a)$ multiplets of $d(a)$ degenerate eigenvalues, where $m(a)$ is given by

$$m(a) = \frac{\omega(a)}{d(a)} = \frac{p!}{a_1!a_2!\dots a_{l-1}!(a_0+a_l)!}\binom{l}{1}^{a_1}\binom{l}{2}^{a_2}\dots\binom{l}{l-1}^{a_{l-1}}. \quad (29)$$

Because the value of E_c depends only on the roots Z_i belonging to the partial packages, we remark that this equation can also be obtained by enumerating the number of choices for these roots: the first factor counts the number of choices for the partial packages among the p packages and the other factors enumerate the choices of i elements among l for each of the a_i partial packages.

In order to know the total number $m(d)$ of multiplets with degeneracy of order d, we must sum over all admissible partitions a with $d(a) = d$,

$$m(d) = \sum_{a;\, d(a)=d} m(a). \tag{30}$$

In the particular case of *half-filling* ($L = 2N$), we have $p = N$ and $l = 2$, $n = 1$. The admissible partitions and the corresponding degeneracies are parameterised by a_0: equation (26) leads to $a_2 = a_0$, $a_1 = N - 2a_0$ and $d(a) = \binom{2a_0}{a_0}$. The relation between the admissible partitions and orders of degeneracies is therefore one-to-one (i.e., two different admissible partitions have different orders of degeneracies): thus the sum in equation (30) reduces to a single term. Explicit formulae and numerical results are given in (Golinelli and Mallick 2004).

For fillings other than 1/2, we have $l > 2$ and two different admissible partitions can lead to the same order of degeneracy. Equation (30) can not be further simplified and the enumeration of admissible partitions seems mandatory. Nevertheless, we can verify the sum rule

$$\Omega = \sum_{d\geq 1} d\; m(d) = \sum_{a} d(a)\; m(a) = \sum_{a} p! \prod_{i=0}^{l} \frac{1}{a_i!} \binom{l}{i}^{a_i} \tag{31}$$

where the last sum runs over the admissible partitions and Ω, defined in equation (1), is the size of the Markov matrix. We use the identity

$$(x+1)^L = \Big[(x+1)^l\Big]^p = \left[\sum_{i=0}^{l} \binom{l}{i} x^i\right]^p = \sum_{a_0,\ldots,a_l} p! \prod_{i=0}^{l} \frac{1}{a_i!} \binom{l}{i}^{a_i} x^{ia_i} \tag{32}$$

where $\sum_i a_i = p$. We remark that Ω is the coefficient of x^N in $(x+1)^L$, whereas the coefficient of x^N on the r.h.s. is precisely the number of admissible partitions defined in equation (26) and is identical to the r.h.s of equation (31).

In Table 1, the explicit example $L = 15$ and $N = 5$ is worked out. We list the admissible partitions, calculate the corresponding order of degeneracy from formula (28) and enumerate the corresponding multiplets by using equation (29).

These results are invariant under 'particle - hole' exchange: the exclusion process with N particles jumping to the right can be mapped to a system with $L - N$ particles jumping to the left after performing a particle - hole exchange. Of course the spectrum of the Markov matrix does not depend on the jumping direction. Thus, we know a priori that the Markov matrices of the TASEP with N and $L - N$ particles have the same spectrum. We

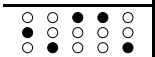
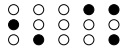
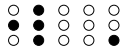
packages	a_0	a_1	a_2	a_3	$d(a)$	$m(a)$
	0	5	0	0	1	243
	1	3	1	0	1	1620
	2	1	2	0	1	810
	2	2	0	1	3	90
	3	0	1	1	4	15

Table 1: An example of calculation of the degeneracies with $L = 15$ and $N = 5$. Each row describes an admissible partition $a = (a_0, \dots, a_l)$ where a_i counts the packages with i chosen roots. In the first column of this table, we have drawn one example of a choice set corresponding to a that selects $N = 5$ roots amongst the $p = 5$ packages of $l = 3$ roots each. The eigenvalues corresponding to a form $m(a)$ multiplets of order of degeneracy $d(a)$, where $d(a)$ and $m(a)$ are given by equations (28, 29). The number of singlets ($d = 1$) is obtained by summing the first three contributions, so 2673 singlets.

now verify this symmetry on the formulae derived above. Denoting with a 'tilde' the quantities for the model with $\tilde{N} = L - N$ particles, we find that $p = \tilde{p}$ (because $\gcd(L, N) = \gcd(L - N, N)$), $\tilde{l} = l$ and $\tilde{n} = l - n$. By a particle - hole exchange, a partition $a = (a_0, \dots, a_l)$ is transformed to $\tilde{a}$ where $\tilde{a_i} = a_{l-i}$. Hence, according to equations (28, 29), we obtain $d(\tilde{a}) = d(a)$ and $m(\tilde{a}) = m(a)$, i.e., degeneracies are indeed invariant by the particle - hole exchange.

6 Discussion

In this section, we draw some consequences of equations (26-30).

We first verify that these equations always predict the existence of singlets (i.e., isolated eigenvalues with degeneracy $d = 1$). In equation (28), we see that $d = 1$ if and only if the partition has no empty package ($a_0 = 0$) or no full package ($a_l = 0$) or if both $a_0 = a_l = 0$. For instance, the stationary eigenvalue 0 is always a singlet thanks to Perron-Frobenius theorem: in fact, the choice set for the stationary state is $c = \{1, \dots, N\}$ and hence each package $\mathcal{P}_k$ has $n = N/p$ selected elements. The corresponding partition is thus given by $a_i = 0$ for $i \neq n$ and $a_n = p$: this implies $d = 1$, as expected.

In order to obtain degenerate eigenvalues i.e., admissible partitions with $d \geq 2$, we must have $a_0 \neq 0$ and $a_l \neq 0$, i.e., at least one empty package and one full package. According to equation (26), the existence of degeneracies

L	N	$m(1)$	$m(2)$	$m(6)$	$m(20)$	$m(70)$
2	1	2				
4	2	4	1			
6	3	8	6			
8	4	16	24	1		
10	5	32	80	10		
12	6	64	240	60	1	
14	7	128	672	280	14	
16	8	256	1792	1120	112	1
18	9	512	4608	4032	672	18

Table 2: *Spectral degeneracies in the TASEP at filling $\rho = 1/2$: L is the size of the lattice, N the number of particles; the other columns give $m(d)$ the number of multiplets with degeneracy d.*

ρ	L	N	$m(1)$	$m(2)$	$m(3)$	$m(4)$	$m(5)$	$m(15)$
1/3	9	3	81		1			
	12	4	459		12			
	15	5	2673		90	15		
	18	6	15849		540	270		1
	21	7	95175		2835	2835	189	21
1/4	16	4	1816			1		
	20	5	15424			20		
	24	6	133456			240	36	
1/5	25	5	53125				1	
2/5	15	6	4975	15				

Table 3: *Examples of spectral degeneracies in the TASEP at filling $\rho \neq 1/2$: L is the size of the lattice, N the number of particles; the other columns give $m(d)$ the number of multiplets with degeneracy d.*

is given by the condition $l \leq N \leq L - l$ or equivalently by

$$L \leq pN \leq pL - L \tag{33}$$

where $p = \gcd(L, N)$. Some numerical examples are given in Tables 2 and 3.

We now analyse the TASEP at a fixed value of the filling $\rho = n/l$, n and l being mutually prime. The two integers (L, N) are parameterised by the single number p defined in equation (21). According to the condition (33), degeneracies appear when

$$p \geq \max\left(\frac{1}{\rho}, \frac{1}{1-\rho}\right) . \tag{34}$$

This condition can always be fulfilled when $0 < \rho < 1$. Moreover, when the system size L and the number of particles N increase with a given rational ρ, higher and higher orders of degeneracies appear and become generic in the large p (or L) limit. The admissible partition $a = (a_0, \ldots, a_l)$ that maximises $\omega(a) = d(a)\, m(a)$ (i.e., the total number of eigenvalues associated with a) is given by

$$a_i \sim p \binom{l}{i} \rho^i (1-\rho)^{l-i} \ . \tag{35}$$

The corresponding order of degeneracy increases exponentially with the size L as $d \propto D^L$ where

$$D = \left(1 + \frac{v}{u}\right)^{u/l} \left(1 + \frac{u}{v}\right)^{v/l} \text{ with } u = \rho^l,\ v = (1-\rho)^l. \tag{36}$$

For example, $D_{1/2} = 2^{1/4} \approx 1.189$ for $\rho = 1/2$; $D_{1/3} = (9^3/2^8)^{1/27} \approx 1.040$ for $\rho = 1/3$, etc... D converges rapidly to 1 when the denominator l of ρ grows. That explains why, in numerical studies of systems of limited size, degeneracies are found only when l is rather small, i.e., when ρ is a 'simple' fraction.

Similarly, we can also determine the admissible partition a that maximises $m(a)$, the number of multiplets. This optimal partition has $a_l = 0$ when $\rho < 1/2$, and $a_0 = 0$ when $\rho > 1/2$. In either case, this corresponds to $d = 1$. Thus the most numerous multiplets for $\rho \neq 1/2$ are singlets. This result does not contradict equations (35, 36) in which the product $d(a)\, m(a)$ is maximised. For the special case, $\rho = 1/2$, the partition that maximises $m(a)$ has both a_0 and a_l different from 0 and the corresponding order of degeneracy increases as $d \propto 2^{L/6}$ (Golinelli and Mallick 2004).

Furthermore, for a given number N of particles, we can search the values of L where degeneracies appear. Because of the particle - hole symmetry (i.e., $N \Leftrightarrow L - N$), we need to consider only the case $N \leq L/2$. Then, the condition (33) becomes $2N \leq L \leq pN$. Because $p \leq N$, this implies that $L \leq N^2$: only a finite number of L values are possible. In the dilute limit (when L becomes large and N remains fixed), the TASEP has thus no degeneracy.

All these results have been derived on the basis of the 'one-to-one' hypothesis stated at the beginning of section 5. It is therefore crucial to compare our formulae with numerical results. We have numerically diagonalised the Markov matrix of the TASEP for certain values of the parameters (L, N). We use the translation symmetry to split the matrix of size Ω into L matrices of size about Ω/L. The spectrum is then computed by using Lapack library (Anderson et al 1999) and degeneracies are counted; details about this

procedure are given in (Golinelli and Mallick 2004). We have investigated systematically all the systems (L, N) with $L \leq 19$. For $L \geq 20$, we have studied the systems (L, N) in which non-trivial degeneracies are predicted and in which the size of the diagonalised matrix remains less than 6000. Results are given in Table 2 for $\rho = 1/2$ and in Table 3 for other values of ρ. The numerical results are in perfect agreement with our analytical predictions, equations (28-30).

The degeneracies for systems much larger than those listed in Tables 2 and 3 can be calculated from the formulae (28-30) for systems with several hundred sites and particles. However, the full numerical diagonalisation of the Markov matrix consumes a computer time of the order of $\Omega^3 \propto (\rho^\rho (1-\rho)^{1-\rho})^{-3L}$. Such a fast growth limits the comparison between numerical diagonalisations and the exact formulae.

7 Conclusion

The spectrum of the Markov matrix of the TASEP on a one-dimensional periodic lattice has a rich structure with multiplets having degeneracies of high order. This structure depends on the filling fraction and presents arithmetical properties related to some particular partitions of the total number of particles. We have derived analytical formulae for the spectral degeneracies by analysing the Bethe equations for the TASEP. These predictions have been verified by numerical calculations and we conjecture that the formulae we propose are exact. Our arguments are based on a 'one-to-one hypothesis' which is stronger than assuming the completeness of the Bethe Ansatz. Although the agreement between numerical results and analytical predictions is a strong argument in favour of the one-to-one hypothesis, it is possible that this hypothesis is not satisfied but that a weaker formulation, leading to the same spectral structure, holds good. We plan to study the completeness of the Bethe Ansatz and this one-to-one hypothesis more precisely in a future work.

Our derivation of the spectral structure from the Bethe equations is rather indirect. In fact, the presence of such high orders of degeneracies is a strong evidence for hidden symmetries in the model. In other words, the TASEP should be invariant under a group operating on the configuration space such that each multiplet of a given order of degeneracy is an irreducible representation of this group. The orders of degeneracies would then classify the dimensions of irreducible representations and the number of multiplets of a given order would represent the multiplicity of a given irreducible representation in the global representation over the configuration space. By using

the techniques of the algebraic Bethe Ansatz, we have constructed a series of nonlocal operators that act on multiple particles and that commute with the Markov matrix. We hope that a study of the representations of these operators will allow us to understand the structure of the spectrum in a purely algebraic manner without having to analyse the solutions of the Bethe equations.

The TASEP evolution can be generalized by introducing a fugacity parameter λ, i.e., by multiplying each non-diagonal term of the Markov matrix by a factor λ. This parameter has been used to calculate the large deviation function of the total particle displacement (Derrida and Lebowitz 1999). We verified that the spectral degeneracies for the 'TASEP + fugacity' model are the same as those found for TASEP. This is not surprising because the properties of the Bethe equations are not altered by introducing a fugacity and the arguments given in sections 4 and 5 can be generalized without any difficulty. However, in the case of the *partially* asymmetric exclusion process, in which particles can jump in both directions, the spectrum has a much simpler structure: eigenvalues are either singlets or doublets. Lastly, for the TASEP on an open lattice, the spectrum is made only of singlets.

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Appendix: Layout of the solutions of the Bethe equations

In this Appendix, we explain how to label the L solutions $(Z_1, \ldots, Z_L)$ of the polynomial equation of degree L

$$(1-Z)^N(1+Z)^{L-N} = Y\,, \tag{37}$$

$(0 \leq N \leq L)$, in such a way that each $Z_k(Y)$ is an analytic function of the parameter Y in the complex plane with a branch cut along the real semi-axis $[0, +\infty)$.

A non-zero complex number Y can be written in a unique way as

$$Y = r^L\, e^{i\phi} \quad \text{with} \quad 0 \leq \phi < 2\pi\,, \tag{38}$$

r being a positive real number. This determination of the argument has a branch cut along $[0, +\infty)$. For a given value of r, the complex numbers Z_k

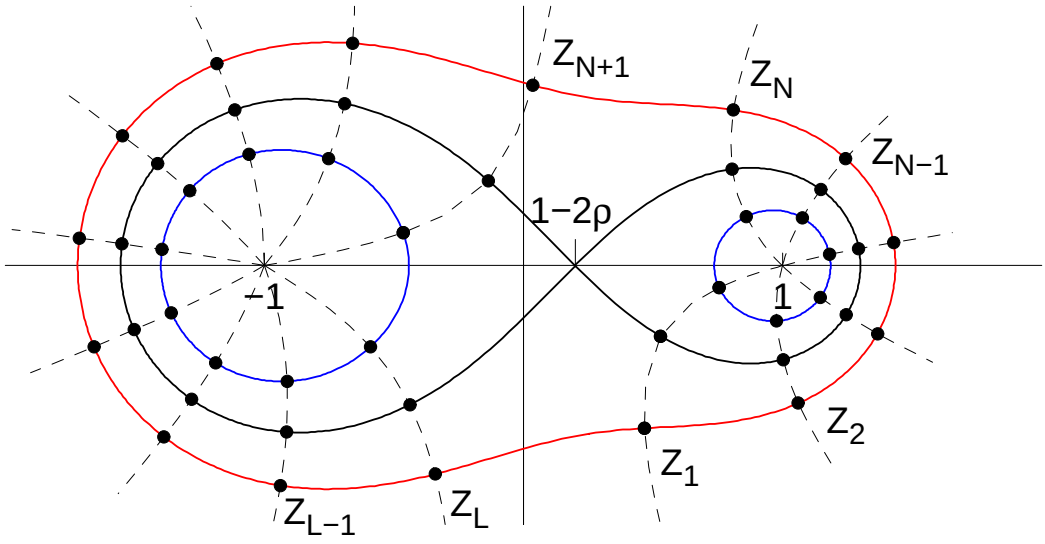

Figure 1: *Labelling the roots of the Bethe equations. Here* $L = 15$, $N = 6$, $\phi = \pi/2$ *and* $r/r_c = 0.8, 1, 1.2$ *(see text). The (resp. blue, black and red) continuous curves are the corresponding Cassini ovals. When* r *is fixed and* ϕ *varies from 0 to* 2π, *each* Z_k *slips counterclockwise along a part of the Cassini oval. When* ϕ *is fixed and* r *varies from* ∞ *to 0, each* Z_k *travels along a dashed curve from* ∞ *to points +1 or -1.*

belong to the generalized Cassini oval defined by

$$|Z-1|^{\rho}|Z+1|^{1-\rho} = r \tag{39}$$

where $\rho = N/L$ is the filling of the system. As shown in Fig. 1, the topology of the Cassini oval depends on the value of r with a critical value

$$r_c = 2\rho^{\rho}(1-\rho)^{1-\rho}\,: \tag{40}$$

- for $r < r_c$, the curve consists of two disjoint ovals with N solutions on the oval surrounding $+1$ and $L-N$ solutions on the oval surrounding -1.
- for $r = r_c$, the curve is a deformed Bernoulli lemniscate with a double point at $Z_c = 1 - 2\rho$.
- for $r > r_c$, the curve is a single loop with L solutions.

The Cassini ovals are symmetrical only if $\rho = 1/2$.

In order to label the solutions, we start by considering the limit $r \to \infty$ for a given ϕ. Equation (37) then becomes

$$Z^L \sim r^L \exp[i(\phi - N\pi)]\,. \tag{41}$$

The solutions Z_k are labelled by

$$Z_k \sim r \, \exp\left[\frac{i}{L}[\phi - N\pi + 2(k-1)\pi]\right] \quad \text{with} \quad k = 1, \ldots, L \, . \tag{42}$$

In other words, the Z_k are regularly distributed along a large circle of radius r with

$$\frac{\phi - N\pi}{L} \leq \arg Z_1 < \ldots < \arg Z_L < \frac{\phi - N\pi}{L} + 2\pi \, . \tag{43}$$

This labelling, obtained for large r, is extended by analytic continuation to all values of r, keeping ϕ *fixed*. The loci of the Z_k are drawn in Fig. 1 (dashed curves): they are orthogonal to the Cassini ovals. A singularity appears along the branch $\phi = 0$ because Z_1 and Z_{N+1} collapse into each other at the double point Z_c when $r = r_c$; we circumvent it by choosing $\phi = 0^+$.

With this labelling, the solutions are ordered along the Cassini ovals. Moreover, when $r < r_c$, the solutions $(Z_1, \ldots, Z_N)$ group together on the right oval and $(Z_{N+1}, \ldots, Z_L)$ on the left oval.

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